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Wilson loops as matrix strings

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Abstract: In the framework of Matrix theory we show that Wilson loops can serve as interpolating fields to define string scattering amplitudes as gauge theory observables.

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1. Introduction

Matrix theory [1] is a proposed formulation of M-theory in the light cone gauge, or for finite N the discrete light cone quantization of M theory [2]. When one of the target space directions is compactified it turns into a non perturbative formulation of light cone type-IIA string theory [3, 4, 5]. In the UV this theory is two dimensional supersymmetric U(N) gauge theory, the dimensional reduction of ten dimensional SYM.

The theory lives on a cylinder parameterized by σ, τ where σ has period 2π. There are two gauge fields A^α, eight scalars X^i ($i = 2, \ldots, 9$) and their superpartners Ψ. The action is

$$\mathcal{S} = \int d\tau d\sigma \, \mathrm{Tr} \left[\frac{1}{4g_{YM}^2} F_{\alpha\beta}^2 + \frac{1}{2} \left(D_\alpha X^i \right)^2 + \frac{g_{YM}^2}{4} [X^i, X^j]^2 + \right.$$
$$\left. + i\bar{\Psi}\Gamma^\alpha D_\alpha \Psi + i g_{YM} \bar{\Psi}\Gamma^i [X^i, \Psi] \right] . \tag{1.1}$$

This action is a non-abelian generalization of the light cone Green-Schwarz string action. In the infra-red $g_{YM} \to \infty$, the potential energy $[X^i, X^j]^2$ is minimized by commuting matrices, so the action reduces to a collection of free strings. The only subtlety is that the holonomy around the cylinder does not have to be trivial. When this happens, instead of N "short" strings one gets fewer "long" strings. In the large-N limit, those long strings become the usual light cone strings.

It was shown [5] that the least irrelevant operator with all the necessary symmetry that can be added to this theory corresponds to a joining/splitting interaction of the strings.

In [6] those interactions were further studied. It was shown that the singular points in light cone string diagrams are smoothed by a non-commutative region. At high energy the string scattering amplitude is dominated by the Gross-Mende surfaces [7]. For those surfaces the interaction point is replaced by an instanton solution of the self-dual YM equations.

In this approach one solves the (self-dual) YM equations for given boundary conditions. The solutions are instantonic configurations carrying a charge in the permutation group S_N relating the incoming arrangement of the string bits to the outgoing one. For gauge invariance it is necessary to sum over all permutations of the N components of the incoming and outgoing strings. Part of this sum, say over the the arrangement of the incoming strings is trivial, but a large part of the sum is non trivial. There will be many distinct configurations (as many as $\sim N!$) which correspond to the same physical process. Each of them is related to a different instanton.

To calculate the scattering amplitude in this way will require a sum over a very large number of different processes. It is possible that the sum is dominated by a small number of saddle points, but we don't know that. Otherwise, this seems like a very difficult problem, perhaps the sum does not converge.

In this paper we try a different approach. Instead of solving the classical equations of motion, whose solutions are not gauge invariant, we find gauge invariant operators which are the non-abelian generalization of string vertex operators. Using those operators, which are certain Wilson loops, we can define a string scattering amplitude as a gauge theory observable. These correlators can be regarded as the non-perturbative string scattering amplitude, and are well defined objects in the gauge theory. Unfortunately, we don't know how to evaluate them, except by reducing back to a calculation like [6].

2. String vertex operators

We will be studying the Wilson loop operator

$$W(p_i, p_-) = \frac{1}{p_-} \,\mathrm{Tr}\, \mathcal{P} \exp\left(i \oint_C \left(A_1 + \frac{p_i}{p_-} X^i\right) d\sigma\right). \tag{2.1}$$

The curve C wraps the world sheet p_- times along a straight line in the σ direction. p_i are arbitrary parameters that couple to the scalars in the loop (they could be functions of σ).

To see what happens to the loop in the IR we should evaluate it for the fields that remain in that limit. The configurations that survive are the long strings, they have a gauge field $A_\sigma(\sigma^\alpha)$ for which $\exp(i\int_0^{2\pi} A_\sigma(\sigma^\alpha)\, d\sigma^1) = U$ is in the Weyl group of $\mathrm{U}(N)$ (a permutation matrix). At every point all the X^i commute, and when parallel transported, also at different points. It is possible to make a large gauge transformation that will set A_σ to zero and will make all the X^i's diagonal, but not single valued. One has to be careful when doing that, though, since the operator (2.1) is invariant only under small gauge transformations and not under large ones. Instead one is left with U^{p_-} in the trace

$$W(p_i, p_-) \to \frac{1}{p_-} \operatorname{Tr}\left[\exp\left(i\int_0^{2\pi p_-} \frac{p_i}{p_-}\tilde{X}^i(\sigma)\,d\sigma\right) U^{p_-}\right], \tag{2.2}$$

where $\tilde{X}^i$ is X^i after it was diagonalized, and it's defined for σ beyond 2π through the twisted boundary conditions.

If U includes a cycle of length p_-, then in that block U^{p_-} is the unit matrix I_{p_-} and $\tilde{X}^i$ are (after reordering)

$$\tilde{X}^i(\sigma) = \begin{pmatrix} x^i(\sigma) & 0 & \cdots & 0 \\ 0 & x^i(2\pi+\sigma) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & x^i(2\pi(p_- - 1)+\sigma) \end{pmatrix}. \tag{2.3}$$

It's clear that $\int_0^{2\pi p_-} \tilde{X}^i(\sigma)$ in that block is proportional to the identity matrix, so the trace of the exponent is just p_- times the exponent. Therefore on this block the Wilson loop reduces to

$$W(p_i, p_-) \to \exp\left(i\int_0^{2\pi p_-} \frac{p_i}{p_-} x^i(\sigma)\,d\sigma\right). \tag{2.4}$$

This is precisely the form of the vertex operator for the light-cone string [8]! If p_i are constants, this is the vertex operator for the graviton.

There is a subtle point when U contains a cycle of length n which divides p_-. Clearly when U contains no such cycles, the Wilson loop vanishes as expected. But as defined above, it will not give zero for configurations with cycles of length that divides p_-.

To eliminate that problem one could recursively define a modified Wilson loop operator with the extra terms subtracted. Alternatively one could approximate p_- by a prime number, which should be possible for large enough matrices. That only leaves the case of cycles of length 1.

Instead, we think that this effect should be regarded as a real artifact of the finite N theory. This is consistent with the observation [6] that the matrix string graviton seems to include both a long string and many short strings. It also fits well with what's known on the nature of the wave function for bound states of N short strings [9]. There too, for non prime N one gets contributions from all the divisors of N.

We just showed that the Wilson-loop (2.1) turns into the corresponding string vertex operator in the infra-red. Since the YM action reduces to the string action the correlation function

$$\langle W_1 \cdots W_n \rangle, \tag{2.5}$$

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with some of the operators inserted at negative infinite time, and some at positive infinite time is the string scattering amplitude in Matrix theory. This is the non-perturbative definition for the scattering amplitude of strings with corresponding momenta.

One should note that the Wilson loops W are *not* the wave function of the matrix string. In particular they have large support on non-commuting configurations. Likewise, those operators are not the only operators one could use. Adding to W any function whose support does not include the "long strings" configurations will not alter this result. Any such modification would work perfectly well as a vertex operator, the difference corresponds to more massive modes that will not modify the calculation when the incoming and outgoing states are sufficiently far apart.

3. Discussion

We showed that the Wilson loop operators (2.1) can serve as interpolating fields for strings. Using them we defined the Matrix string scattering amplitude as a gauge invariant observable (2.5). One would like to find a way to evaluate those correlators.

This turns out to be a difficult problem, those Wilson loops are not simple operators. If one tries to use perturbation theory, it is natural to expand about classical solutions — the long strings configurations. This reduces the problem back to the calculation of [6]. One has to sum over many ingoing and outgoing classical solutions, and calculate the instantonic processes that take one to the other.

We have not found any other way to calculate these correlators.

Apart from strings, matrix theory contains other objects [10]. It is easy to write down operators which have the appropriate charges. For example, the Wilson loop

$$W = \mathrm{Tr}\,\mathcal{P} \oint_C d\sigma\, F_{01} \exp\left(i \oint_C \left(A_1 + \frac{p_i}{p_-} X^i\right) d\sigma\right), \tag{3.1}$$

can be used as a vertex operator for a 0-brane.

Likewise

$$W = \mathrm{Tr}\ \mathcal{P} \oint_C d\sigma [X^i, X^j] \exp\left(i \oint_C \left(A_1 + \frac{p_i}{p_-} X^i\right) d\sigma\right), \tag{3.2}$$

is a Wilson-loop for a 2-brane in the i, j plane.

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